Canonical Form and Separability of PPT States on Multiple Quantum Spaces

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Abstract By using the "subtracting projectors" method in proving the separability of PPT states on multiple quantum spaces, we derive a canonical form of PPT states in $\mathbb{C}^{K_1} \otimes \mathbb{C}^{K_2} \otimes \cdots \otimes \mathbb{C}^{K_m} \otimes \mathbb{C}^N$ composite quantum systems with rank N, from which a sufficient separability condition for these states is presented.

As the key resource in quantum information processing [1], quantum entanglement has resulted in the explosion of interest in quantum computing and communication in recent years [2]. The separability of quantum pure states is well understood [3]. However for mixed states, although there are already many results related to the separability criterion, the physical character and mathematical structure of the quantum entanglement are still far from being satisfied. The PPT (positive partial transpose) criteria play very important roles in the investigation of separability. It is of significance to investigate the general form of PPT states to study the separability and bound entangled states. In [4] the PPT (mixed) states on $\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^N$ with rank N has been studied and a separability criterion has been obtained. These results have been generalized to the case of PPT states on $\mathbb{C}^2 \times \mathbb{C}^3 \times \mathbb{C}^N$ with rank N [5], and the case of high dimensional tripartite systems [6] and $\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^N$ [7]. In this note we summarize our efforts in striving at some understanding of the properties of quantum entanglement for composite systems. We generalize the results to the most general case of the canonical form of PPT states in $\mathbb{C}^{K_1} \otimes \mathbb{C}^{K_2} \otimes \cdots \otimes \mathbb{C}^{K_m} \otimes \mathbb{C}^N$ composite quantum systems with rank N, and study the separability condition for these states in terms of the canonical form.

Let $\mathbb{C}^K$ be K-dimensional complex vector space with orthonormal basis $\{|\mathbf{i}\rangle\}$, $i = 0, ..., K-1$. A general pure state on $\mathbb{C}^{K_1} \otimes \mathbb{C}^{K_2} \otimes \ldots \otimes \mathbb{C}^{K_m}$ is of the form

$$|\Psi\rangle = \sum_{i=0}^{K_1-1} \sum_{j=0}^{K_2-1} \ldots \sum_{k=0}^{K_m-1} a_{ij\ldots k} |i, j, \ldots, k\rangle, \tag{1}$$

where $|i,j,\ldots,k\rangle = |i\rangle \otimes |j\rangle \otimes \ldots \otimes |k\rangle$, $a_{ij\ldots k} \in \mathbb{C}$, $\sum a_{ij\ldots k} a^*_{ij\ldots k} = 1$ ($*$ denoting complex conjugation). $|\Psi\rangle$ is said to be (fully) separable if $a_{ij\ldots k} = a_i a_j \ldots a_k$ for some $a_i,\ a_j,\ \ldots,\ a_k \in \mathbb{C}$. A mixed state on $\mathbb{C}^{K_1} \otimes \mathbb{C}^{K_2} \otimes \ldots \otimes \mathbb{C}^{K_m}$ is described by a density matrix ρ,

$$\rho = \sum_{i=1}^{M} p_i |\Psi_i\rangle\langle\Psi_i|, \tag{2}$$

for some $M \in I\!\!N$, $0 < p_i \leq 1$, $\sum_{i=1}^{M} p_i = 1$, $|\Psi_i\rangle$s are pure states of the form (1) and $\langle\Psi_i|$ is the transpose and conjugation of $|\Psi_i\rangle$. We call state ρ PPT if $\rho^{T_l} \geq 0$, $\forall l$, where ρ^{T_l} is the transpose of ρ with respect to the l-th subspace.

In the following we denote by $R(\rho)$, $K(\rho)$, $r(\rho)$ and $k(\rho)$ the range, kernel, rank, dimension of the kernel of ρ respectively, where, by definition $K(\rho) = \{|\phi\rangle : \rho|\phi\rangle = 0\}$, $R(\rho) = \{|\phi\rangle : \exists|\psi\rangle$, such that $|\phi\rangle = \rho|\psi\rangle\}$.

We consider now composite quantum systems in $\mathbb{C}^{K_1}_{\mathrm{A}_1} \otimes \mathbb{C}^{K_2}_{\mathrm{A}_2} \otimes \cdots \otimes \mathbb{C}^{K_m}_{\mathrm{A}_m} \otimes \mathbb{C}^{N}_{\mathrm{A}_{m+1}}$ with $r(\rho) = N$, where A_i denotes the i-th subsystem, K_i stands for the dimension of the i-th complex vector space, $m,\ N \in I\!\!N$.

We first derive a canonical form of PPT states in $\mathbb{C}^{2}_{\mathrm{A}_1} \otimes \mathbb{C}^{2}_{\mathrm{A}_2} \otimes \cdots \otimes \mathbb{C}^{2}_{\mathrm{A}_m} \otimes \mathbb{C}^{N}_{\mathrm{A}_{m+1}}$ with rank N, which allows for an explicit decomposition of a given state in terms of convex sum of projectors on product vectors. Let $|0_{\mathrm{A}_1}\rangle$, $|1_{\mathrm{A}_1}\rangle$; $|0_{\mathrm{A}_2}\rangle$, $|1_{\mathrm{A}_2}\rangle$; $\cdots$; $|0_{\mathrm{A}_m}\rangle$, $|1_{\mathrm{A}_m}\rangle$ and $|0_{\mathrm{A}_{m+1}}\rangle\ ,\cdots\ ,|(N-1)_{\mathrm{A}_{m+1}}\rangle$ be some local bases of the sub-systems $\mathrm{A}_1,\ \mathrm{A}_2,\cdots,\ \mathrm{A}_m$, and A_{m+1} respectively. In terms of the method used in [4, 5, 6, 7], we have

Lemma 1. Every PPT state ρ in $\mathbb{C}^{2}_{\mathrm{A}_1} \otimes \mathbb{C}^{2}_{\mathrm{A}_2} \otimes \cdots \otimes \mathbb{C}^{2}_{\mathrm{A}_m} \otimes \mathbb{C}^{N}_{\mathrm{A}_{m+1}}$ such that

$$r(\langle 1_{A_1}, 1_{A_2}, \cdots, 1_{A_m}|\rho|1_{A_1}, 1_{A_2}, \cdots, 1_{A_m}\rangle) = r(\rho) = N,$$

can be transformed into the following canonical form by using a reversible local operation:

$$\rho = \sqrt{F} T^\dagger T \sqrt{F}, \tag{3}$$

where $T = (D_m \quad I) \otimes (D_{m-1} \quad I) \otimes \cdots \otimes (D_1 \quad I)$, D_i, F and the identity I are $N \times N$ matrices acting on $\mathbb{C}^{N}_{\mathrm{A}_{m+1}}$ and satisfy the following relations: $[D_i,\ D_j] = [D_i,\ {D_j}^\dagger] = 0$, and $F = F^\dagger$ ($\dagger$ stands for the transpose and conjugate), $i,j = 1,2,\cdots\ m$.

Using Lemma 1 we can prove the the following Theorem:

Theorem 1. A PPT-state ρ in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \otimes \mathbb{C}^N$ with $r(\rho) = N$ is separable if there exists a product basis $|e_{A_1},\ e_{A_2}, \cdots,\ e_{A_m}\rangle$ such that

$$r(\langle e_{A_1},\ e_{A_2}, \cdots,\ e_{A_m}|\rho|e_{A_1},\ e_{A_2}, \cdots,\ e_{A_m}\rangle) = N$$

.

Proof. According to Lemma 1 the PPT state ρ can be written as the form of (3). Since all the D_i and $D_j^\dagger$ commute, they have common eigenvectors $|f_n\rangle$. Let a_1^n, a_2^n, $\cdots$, a_m^n be the corresponding eigenvalues of $D_1,\ D_2,\ \cdots,\ D_m$ respectively. We have

$$\begin{aligned}\langle f_n|\rho|f_n\rangle &= \left[\begin{pmatrix} a_m^{n\,*} \\ 1 \end{pmatrix} \otimes \begin{pmatrix} {a_{m-1}^n}^* \\ 1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} a_1^{n*} \\ 1 \end{pmatrix}\right] (a_m^n\ \ 1) \otimes (a_{m-1}^n\ \ 1) \otimes \cdots \otimes (a_1^n\ \ 1) \\ &= |e_{A_1},\ e_{A_2}, \cdots,\ e_{A_m}\rangle\langle e_{A_1},\ e_{A_2}, \cdots,\ e_{A_m}|.\end{aligned}$$

We can thus write ρ as

$$\rho = \sum_{n=1}^{N} |\psi_n\rangle\langle\psi_n| \otimes |\phi_n\rangle\langle\phi_n| \otimes \cdots \otimes |\omega_n\rangle\langle\omega_n| \otimes |f_n\rangle\langle f_n|,$$

where

$$|\psi_n\rangle = \begin{pmatrix} a_m^{n\,*} \\ 1 \end{pmatrix}, \quad |\phi_n\rangle = \begin{pmatrix} {a_{m-1}^n}^* \\ 1 \end{pmatrix}, \quad \cdots, \quad |\omega_n\rangle = \begin{pmatrix} a_1^{n*} \\ 1 \end{pmatrix}.$$

Because the local transformations are reversible, we can now apply the inverse transformations and obtain a decomposition of the initial state ρ in a sum of projectors onto product vectors. This proves the separability of ρ. □

By using Lemma 1, Theorem 1 and the method in [6], we can generalize the results to multipartite quantum systems in $\mathbb{C}^{K_1}_{\mathrm{A}_1} \otimes \mathbb{C}^{K_2}_{\mathrm{A}_2} \otimes \cdots \otimes \mathbb{C}^{K_m}_{\mathrm{A}_{\mathrm{M}}} \otimes \mathbb{C}^{N}_{\mathrm{A}_{\mathrm{M}+1}}$ with rank N. Let $|0_{A_i}\rangle$, $|1_{A_i}\rangle$, $\cdots$, $|(K_i-1)_{A_i}\rangle$, $i = 1, 2, \cdots, m$; and $|0_{A_{m+1}}\rangle$ $, \cdots$ $, |(N-1)_{A_{m+1}}\rangle$ be some local bases of the sub-systems A_i, $i = 1, 2, \cdots, m,\ A_{m+1}$ respectively.

Lemma 2. Every PPT state ρ in $\mathbb{C}^{K_1}_{\mathrm{A}_1} \otimes \mathbb{C}^{K_2}_{\mathrm{A}_2} \otimes \cdots \otimes \mathbb{C}^{K_m}_{\mathrm{A}_m} \otimes \mathbb{C}^{N}_{\mathrm{A}_{m+1}}$ such that $r(\langle (K_1-1)_{A_1}, \cdots, (K_m-1)_{A_m}|\rho|(K_1-1)_{A_1}, \cdots, (K_m-1)_{A_m}\rangle) = r(\rho) = N$, can be transformed into the following canonical form by using a reversible local operation:

$$\rho = \sqrt{F} T^\dagger T \sqrt{F}, \tag{4}$$

where $T = (D^1_{K_1-1}\ \cdots\ D^1_1\ \ I) \otimes (D^2_{K_2-1}\ \cdots\ D^2_1\ \ I) \otimes \cdots \otimes (D^m_{K_{m}-1}\ \cdots\ D^m_1\ \ I)$, D^t_i, F and the identity I are $N \times N$ matrices acting on $\mathbb{C}^N_{\mathrm{A}_{m+1}}$ and satisfy the following relations: $[D^t_{i_t},\ D^s_{j_s}] = [D^p_{i_p},\ {D^q_{j_q}}^\dagger] = 0$, and $F = F^\dagger$, $i_t, j_t = 1, 2, \cdots\ K_t,\ t, s, p, q = 1, 2, \cdots\ m$.

Proof. We prove the lemma by induction on m. It is already proved for the cases $m = 1$ [8] and $m = 2$ [6].

Now we consider the case of general m. Suppose that for the case $m-1$ the result is correct. In the considered basis a density matrix ρ can be always written as: $S \times S$-partitioned matrix, where $S = K_1 K_2 \cdots K_m$, with the i-row j-column entry E_{ij}, $r(E_{S\times S}) = N$.

The projection $\langle (K_1-1)_{A_1}|\rho|(K_1-1)_{A_1}\rangle$ gives rise to a state $\tilde{\rho}=\langle (K_1-1)_{A_1}|\rho|(K_1-1)_{A_1}\rangle$ which is a state in $\mathbb{C}^{K_2}_{\mathrm{A}_2}\otimes\cdots\otimes\mathbb{C}^{K_m}_{\mathrm{A}_m}\otimes\mathbb{C}^{N}_{\mathrm{A}_{m+1}}$ with $r(\tilde{\rho})=r(\rho)=N$. The fact that ρ is PPT implies that $\tilde{\rho}$ is also PPT, $\tilde{\rho}\geq 0$. By induction hypothesis , we have

$$\tilde{\rho}=\sqrt{F}{T_1}^{\dagger}T_1\sqrt{F}, \tag{5}$$

where $T_1=(D^2_{K_2-1}\ \cdots\ D^2_1\ \ I)\otimes\cdots\otimes(D^m_{K_{m-1}}\ \cdots\ D^m_1\ \ I)$, with $[D^t_{i_t},\ D^s_{j_s}]=[D^p_{i_p},\ {D^q_{j_q}}^{\dagger}]=0$, $i_t,j_t=1,2,\cdots,\ K_t$, $t,s,p,q=2,\cdots,\ m$.

Similarly, if we consider the projection $\langle (K_2-1)_{A_2}|\rho|(K_2-1)_{A_2}\rangle$, $\cdots$, $\langle (K_m-1)_{A_m}|\rho|(K_m-1)_{A_m}\rangle$, by induction hypothesis, we have $m-1$ relations like (5). In fact, the projection $\langle (K_j-1)_{A_j}|\rho|(K_j-1)_{A_j}\rangle$, $j=2,\cdots,m$, is

$$\rho_j=\sqrt{F}{T_j}^{\dagger}T_j\sqrt{F}, \tag{6}$$

where

$$\begin{aligned}T_j=&(D^1_{K_1-1}\ \cdots\ D^1_1\ \ I)\otimes\cdots\otimes(D^{j-1}_{K_{j-1}-1}\ \cdots\ D^{j-1}_1\ \ I)\\&\otimes(D^{j+1}_{K_{j+1}-1}\ \cdots\ D^{j+1}_1\ \ I)\otimes\cdots\otimes(D^m_{K_{m-1}}\ \cdots\ D^m_1\ \ I),\end{aligned}$$

with $[D^t_{i_t},\ D^s_{j_s}]=[D^p_{i_p},\ {D^q_{j_q}}^{\dagger}]=0$, $i_t,j_t=1,2,\cdots,K_t$, $t,s,p,q=1,\cdots,j-1,j+1,\cdots\ m$. Taking into account all these projections and using the kernel vectors of ρ we can determine part of the entries E_{ij}. Then by using the PPT property of the partial transpose of ρ related to the subsystems: the kernel vectors of ρ are also the kernel vectors of the partial transpose of ρ; if E_{jj} is known, the rest entries of ρ, E_{ij}, $i<j$, are also determined. For the diagonal elements like E_{11}, we define $\Delta=E_{11}-S^{\dagger}S$, where $S=D^1_{K_1-1}D^2_{K_2-1}\cdots D^m_{K_m-1}$, to determine E_{11} by proving that $\Delta=0$. It is straightforward to prove that there exist matrices $D^i_{s_i}$, $s_i=1,2,\cdots,K_i-1$, $i=1,2,\cdots,m$ satisfying the relation (4). $\square$

From Lemma 2 we have the following conclusion:

Theorem 2. A PPT-state ρ in $\mathbb{C}^{K_1}_{\mathrm{A}_1}\otimes\mathbb{C}^{K_2}_{\mathrm{A}_2}\otimes\cdots\otimes\mathbb{C}^{K_m}_{\mathrm{A}_m}\otimes\mathbb{C}^{N}_{\mathrm{A}_{m+1}}$ with $r(\rho)=N$ is separable if there exists a product basis $|e_{A_1},\ e_{A_2},\cdots,\ e_{A_m}\rangle$ such that

$$r(\langle e_{A_1},\ e_{A_2},\cdots,\ e_{A_m}|\rho|e_{A_1},\ e_{A_2},\cdots,\ e_{A_m}\rangle)=N$$

.

We have derived a canonical form of PPT states in $\mathbb{C}^{K_1}\otimes\mathbb{C}^{K_2}\otimes\cdots\otimes\mathbb{C}^{K_m}\otimes\mathbb{C}^{N}$ composite quantum systems with rank N, together with a sufficient separability criterion from this canonical form. Generally the separability criterion we can deduce here is weaker than in the cases of $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^N$ and $\mathbb{C}^2\otimes\mathbb{C}^3\otimes\mathbb{C}^N$, as the PPT criterion is only sufficient and necessary for the separability of bipartite states on $\mathbb{C}^2\otimes\mathbb{C}^2$ and $\mathbb{C}^2\otimes\mathbb{C}^3$. Besides the

discussions of separability criterion, the canonical representation of PPT states can shade light on studying the structure of bound entangled states. One can check if these PPT states are bound entangled by checking whether they are entangled or not.

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